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Rank dependency of weight multiplicities for simple Lie groups

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Abstract. The determination of the rank dependence of the weight multiplicities for the groups O_{2k+1}, Sp_{2k} and O_{2k} is simplified using the properties of Schur functions together with a simple weight space mapping. The weight multiplicities are factorised into products of much simpler quantities whose rank dependence is readily computed. The spinor weight multiplicities for SO_{2k} are shown to be simply related to those of O_{2k}. The weight multiplicities for the k-part tensor representation of SO_{2k} are examined in terms of difference characters. Some specific results are tabulated for SO_{2k}.

1. Introduction

Many algorithms have been developed for determining the weight multiplicities for irreducible representations (irreps) of the compact semi-simple Lie groups (Weyl 1926, Freudenthal 1954, Konstant 1959, Racah 1964, Speiser 1964). These algorithms generally fail to indicate the rank dependence of the weight multiplicities. King and Plunkett (1976) have indicated how the properties of Schur functions ($\mathcal{S}$ functions) may be exploited to give the rank dependence of weight multiplicities for the classical Lie groups. They give specific results for the groups U_k, O_{2k+1}, Sp_{2k}. In this paper, I show how the evaluation of the rank dependence of the weight multiplicities for the groups O_{2k+1}, Sp_{2k} and O_{2k} may be simplified, and more importantly, treat the special case of SO_{2k} for the first time. Detailed results are presented for the latter case. I draw heavily upon King and Plunkett (1976) for matters of notation.

2. Weight multiplicities for U_k

The unitary group U_k admits a maximal toroidal subgroup

$$\mathcal{T}_k \sim U_1^1 \times U_1^2 \times \ldots \times U_1^k \tag{1}$$

which is the product of k one-dimensional unitary groups U_1. The irreps of the U_1 groups are all one-dimensional and may be labelled as $\{\omega\}$ where ω is a positive or negative real number such that

$$\{\bar{\omega}\} = \{-\omega\} \tag{2}$$

where $\{\bar{\omega}\}$ is contragredient to $\{\omega\}$.

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The irreps of $\mathcal{T}_k$ are all one-dimensional and may be specified as

$$\{\boldsymbol{\omega}\} = \{\omega_1\}\{\omega_2\} \ldots \{\omega_k\}$$

or for brevity as

$$\{\boldsymbol{\omega}\} = \{\omega_1\omega_2 \ldots \omega_k\}. \tag{3}$$

The k numbers ω_i may be regarded as the k components of a vector in a k-dimensional vector space with $(\boldsymbol{\omega})$ being a *weight vector.* Since $\mathrm{U}_k \supset \mathcal{T}_k$ we have for the covariant irreps $\{\lambda\}$ of U_k under $\mathrm{U}_k \downarrow \mathcal{T}_k$

$$\{\lambda\} \downarrow M_{\{\lambda\}}^{(\boldsymbol{\omega})}(\boldsymbol{\omega}) \tag{4}$$

where $M_{\{\lambda\}}^{(\boldsymbol{\omega})}$ is the *weight multiplicity* of the weight $(\boldsymbol{\omega})$ in the reduction of $\{\lambda\}$. Since the irreps of $\mathcal{T}_k$ are all one-dimensional the number of weights $(\boldsymbol{\omega})$ obtained in (4) must equal the dimension of the irrep $\{\lambda\}$ of U_k.

As noted by King and Plunkett (1976) the weight multiplicities $M_{\{\lambda\}}^{(\boldsymbol{\omega})}$ of U_k are k independent and are in one-to-one correspondence with the integers $K_{\{\lambda\}}^{(\boldsymbol{\omega})}$ associated with the Kostka matrix (cf Macdonald 1979). For our purpose we shall assume the $M_{\{\lambda\}}^{(\boldsymbol{\omega})}$ are known for U_k.

3. Dominant weights

The weights $(\boldsymbol{\omega})$ of an irrep of some Lie group G exhibit reflection symmetry under the elements $\mathcal{S}$ of the Weyl group $\mathrm{W_G}$ (cf King and Al-Qubanchi 1981). Given a weight vector $(\boldsymbol{\omega})$ of G then the vectors in the set $\{\mathcal{S}\omega : \mathcal{S} \in \mathrm{W_G}\}$ are said to be G equivalent to $(\boldsymbol{\omega})$. The highest weight vector of a G equivalent set is said to be the G-*dominant weight vector.* Once the dominant weights, and their multiplicities, are known the remaining weights follow simply from the operations $\mathcal{S}$ of $\mathrm{W_G}$.

Let

$$\binom{k}{s_1\, s_2 \ldots} = \frac{k!}{(k-u)!\, \Pi\, s_i!} \tag{5}$$

with $u = \Sigma\, s_i$. Then if $(\boldsymbol{\omega}) = \omega_1^{s_1}\omega_{s_1+1}^{s_2} \ldots$ is a dominant weight of U_k, O_{2k+1}, Sp_{2k} or O_{2k} the number of weights connected to $(\boldsymbol{\omega})$ by the Weyl symmetry operations of the appropriate Weyl group is given by

$$\mathrm{U}_k \qquad n(\boldsymbol{\omega}) = \binom{k}{s_1\, s_2 \ldots} \tag{6a}$$

$$\mathrm{O}_{2k+1}, \mathrm{Sp}_{2k}, \mathrm{O}_{2k} \qquad n(\boldsymbol{\omega}) = 2^u \binom{k}{s_1\, s_2 \ldots} \tag{6b}$$

for the tensor irreps. In the case of the spinor irreps $[\Delta; \lambda]$ of O_{2k+1} and O_{2k} the number of weights connected is

$$n(\boldsymbol{\omega}) = n(\Delta; \boldsymbol{\sigma}) = 2^k \binom{k}{s_1\, s_2 \ldots} \tag{6c}$$

where $(\Delta; \boldsymbol{\sigma})$ denotes weight vectors of the form

$$(\sigma_1+\tfrac{1}{2}, \sigma_2+\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}).$$

In the case of SO_{2k+1} equations (6*b*) and (6*c*) remain unchanged while for SO_{2k} the right-hand side of (6*b*) must be halved if $\omega_k \neq 0$ and in every case for (6*c*). This follows since the irreps $[\lambda]$ (tensor or spinor) of O_{2k} involving $\lambda_k \neq 0$ split under $O_{2k}\downarrow SO_{2k}$ into conjugate pairs of irreps of SO_{2k} and the weight space splits into two equal portions with half the weights of $[\lambda]$ of O_{2k} in one and the other half in the other. The action of the Weyl group for SO_{2k} is such that dominant weights having k non-zero parts with an *even*(*odd*) number of minus signs are connected to weights also having an *even*(*odd*) number of minus signs. Weights of different sign parity are totally disjoint.

4. Weight space mappings

The weight multiplicities $M_{\{\lambda\}}^{(\omega)}$ of the covariant irreps of U_N are N independent and the weights $(\boldsymbol{\omega})$ span a N-dimensional weight space. The weights $(\boldsymbol{\sigma})$ of the classical groups O_{2k+1}, Sp_{2k} and O_{2k} all span a k-dimensional weight space. The weight multiplicities for these groups are generally k dependent.

The set of weights $\{(\boldsymbol{\omega})\}$ of some irrep of U_N may be used to form a set of weights $\{(\boldsymbol{\sigma})\}$ under the mapping $\mathscr{P}$

$$\omega_i - \omega_{N+1-i} \xrightarrow{\mathscr{P}} \sigma_i \qquad i = 1, 2, \ldots, k \tag{7}$$

where $N = 2k$ or $2k+1$. In general the mapping $\mathscr{P}$ may be expressed in terms of dominant weights $(\boldsymbol{\sigma})$ of the k-dimensional subspace of $(\boldsymbol{\omega})$ by writing

$$(\boldsymbol{\omega}) \xrightarrow{\mathscr{P}} \mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})}(\boldsymbol{\sigma}) \tag{8}$$

where $\mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})}$ is the multiplicity with which the dominant weight $(\boldsymbol{\sigma})$ occurs. These multiplicity numbers are not necessarily k independent.

The array of numbers $\mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})}$ for fixed ω_ω $(\omega_\omega = \Sigma\, \omega_i)$ is dominated by null entries. These certainly arise if $p_\sigma > p_\omega$ or $p_\sigma = p_\omega$ *and* $\omega_\sigma \neq \omega_\omega$, or $\boldsymbol{\omega} - \boldsymbol{\sigma}$ results in any negative weight components. (NB we use p_κ to stand for the number of parts of (κ) etc.) Furthermore,

$$\mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})} = \delta_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})} \qquad \text{if} \quad \omega_\omega = \omega_\sigma \tag{9}$$

while

$$\mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})} = N(\mathrm{mod}\, 2) \qquad \text{if} \quad \omega_\omega = \omega_\sigma + 1. \tag{10}$$

Of the remaining entries for $\mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})}$ some will be explicitly k dependent, some will depend on the parity of N, and some will be equal to an integer that is independent of N or k. In all cases it is a trivial combinatorial task to compute the k dependence of $\mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})}$ for arbitrary $(\boldsymbol{\omega})$ and $(\boldsymbol{\sigma})$. A given array of $\mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})}$ is checked by noting that

$$n(\boldsymbol{\omega}) = \mathscr{M}_{(\boldsymbol{\omega})}^{(\boldsymbol{\sigma})} n(\boldsymbol{\sigma}) \tag{11}$$

where $n(\boldsymbol{\omega})$ and $n(\boldsymbol{\sigma})$ are given by (6*a*) and (6*b*) respectively.

5. A k-dependent unitary group mapping

A k-dependent multiplicity number $M_{(\lambda)}^{(\sigma)}$ may be defined by writing

$$M_{\{\lambda\}}^{(\sigma)} = K_{\{\lambda\}}^{(\omega)} \mathcal{M}_{(\omega)}^{(\sigma)} \tag{12}$$

where $(\boldsymbol{\sigma})$ is a weight for a k-dimensional weight space associated with the covariant irrep $\{\lambda\}$ of U_N. The multiplicities $M_{\{\lambda\}}^{(\sigma)}$ may be readily calculated from a knowledge of $\mathcal{M}_{(\omega)}^{(\sigma)}$ multiplicities just derived and those of the Kostka matrix $K_{\{\lambda\}}^{(\omega)}$.

6. Weight multiplicities for O_{2k+1}, O_{2k} and Sp_{2k}

The weight multiplicities $M_{[\lambda]}^{(\sigma)}$ and $M_{\langle\lambda\rangle}^{(\sigma)}$ associated with O_N and Sp_{2k} respectively may be calculated from a knowledge of the k-dependent multiplicities just considered by noting that

$$M_{[\lambda]}^{(\sigma)} = M_{\{\lambda/C\}}^{(\sigma)} \tag{13a}$$

and

$$M_{\langle\lambda\rangle}^{(\sigma)} = M_{\{\lambda/A\}}^{(\sigma)} \tag{13b}$$

where A and C are $\mathcal{S}$-function series (King 1975).

7. Weight multiplicities for spinor irreps of O_N

We now turn to the evaluation of weight multiplicities $M_{[\Delta;\lambda]}^{(\Delta;\sigma)}$ for the spinor irreps of O_N. We start by noting that (King 1975)

$$[\Delta; \lambda] = \Delta \cdot [\lambda/P]. \tag{14}$$

We then have for the tensor irreps in $[\lambda/P]$ the weights

$$M_{[\lambda/P]}^{(\tau)}(\boldsymbol{\tau}) \tag{15}$$

where the multiplicities $M_{[\mu]}^{(\tau)}$ follow from the previous section. The 2^k weights associated with the basic spinor irrep Δ of O_N ($N = 2k$ or $2k+1$) are all of the form

$$(\Delta) = \underbrace{(\pm\tfrac{1}{2} \pm\tfrac{1}{2} \ldots \pm\tfrac{1}{2})}_{k \text{ terms}} \tag{16}$$

where all possible combinations of signs are to be taken. These weights (Δ) must be combined additively with the weights $(\boldsymbol{\tau})$ to give

$$\Delta(\boldsymbol{\tau}) {\downarrow} \mathcal{M}_{\Delta(\tau)}^{(\Delta;\sigma)}(\Delta; \boldsymbol{\sigma}). \tag{17}$$

Combining (15) and (17) gives the weights for the irrep $[\Delta; \lambda]$ of O_N as

$$[\Delta; \lambda] {\downarrow} M_{[\lambda/P]}^{(\tau)} \mathcal{M}_{\Delta(\tau)}^{(\Delta;\sigma)}(\Delta; \boldsymbol{\sigma}) \tag{18}$$

and hence the weight multiplicities are given as

$$M_{[\Delta;\lambda]}^{(\Delta;\sigma)} = M_{[\lambda/P]}^{(\tau)} \mathcal{M}_{\Delta\cdot(\tau)}^{(\Delta;\sigma)}. \tag{19}$$

The multiplicities $\mathcal{M}^{(\Delta;\sigma)}_{\Delta\cdot(\tau)}$ can be readily evaluated in a k-dependent form by considering weights of the form

$$(\tau_1 \pm \tfrac{1}{2}, \lambda_2 \pm \tfrac{1}{2}, \ldots, \tau_k \pm \tfrac{1}{2}) \tag{20}$$

together with the operations of the Weyl group W_{O_N}. Thus to evaluate $M^{(\Delta;1)}_{\Delta\cdot(1^2)}$ we note that the dominant weights will be of the form

$$(1+\tfrac{1}{2}, 1-\tfrac{1}{2}, \pm\tfrac{1}{2}, \ldots, \pm\tfrac{1}{2}).$$

The second unit can occur in any of $k-1$ places and hence

$$\mathcal{M}^{(\Delta;1)}_{\Delta\cdot(1^2)} = k-1.$$

We note that non-zero entries in the arrays $\mathcal{M}^{(\Delta;\sigma)}_{\Delta(\tau)}$ are sparse. Consideration of the weights in $\Delta\cdot(\tau)$ shows that the weight $(\Delta;\sigma)$ arises if and only if

$$\{\tau/1^s\} \supset \{\sigma\}. \tag{21}$$

8. Weight multiplicities for spinor irreps of SO_{2k}

The spinor irreps $[\Delta;\lambda]$ of O_{2k} split into a pair of conjugate irreps under $O_{2k}\downarrow SO_{2k}$ to give

$$[\Delta;\lambda]\downarrow[\Delta;\lambda]_+ + [\Delta;\lambda]_-. \tag{22}$$

Noting that (King *et al* 1981)

$$[\Delta;\lambda]_\pm = \sum_m (-1)^m\, \Delta_{\pm(-)^m}[\lambda/m] \tag{23}$$

leads to

$$M^{(\Delta;\sigma)_{(-)^x}}_{[\Delta;\lambda]_+} = \sum_m (-1)^m M^{(\tau)}_{[\lambda/m]} \mathcal{M}^{(\Delta;\sigma)_{(-)^x}}_{\Delta_{(-)^m}\cdot(\tau)}. \tag{24}$$

The terms in $\Delta_{(-)^m}\cdot(\tau)$ are all of the form

$$(\Delta;\omega)_{(-)^k}$$

where $k = \omega_\tau - \omega_\omega + m$. However, the terms in $[\lambda/m]$ must satisfy

$$\omega_\tau = \omega_\lambda - m - 2s$$

where s is an integer and hence

$$k = \omega_\lambda - \omega_\omega - 2s. \tag{25}$$

As a consequence of (24) we must have

$$x = (\omega_\lambda - \omega_\sigma) \bmod 2. \tag{26}$$

Thus we arrive at the important conclusion that

$$M^{(\Delta;\sigma)_{\pm(-)^x}}_{[\Delta;\lambda]_\pm} = M^{(\Delta;\sigma)}_{[\Delta;\lambda]} \tag{27}$$

where x satisfies (26) and we have related the multiplicities for SO_{2k} to those of O_{2k} in (27).

9. Weight multiplicities for tensor irreps of SO_{2k}

If $[\lambda]$ is a tensor irrep of SO_{2k} with $\lambda_k=0$ then the weight multiplicities for SO_{2k} are identical to those for O_{2k}. However, if $\lambda_k\neq 0$ then under $O_{2k}\downarrow SO_{2k}$ we have (King *et al* 1981)

$$[\lambda]\downarrow[\lambda]_+ + [\lambda]_- \qquad \lambda_k\neq 0. \tag{28}$$

If $(\boldsymbol{\sigma})$ is a weight of O_{2k} with $\sigma_k=0$ then for SO_{2k}

$$M^{(\boldsymbol{\sigma})}_{[\lambda]_\pm}=\tfrac{1}{2}M^{(\boldsymbol{\sigma})}_{[\lambda]} \tag{29}$$

and the weight multiplicities also follow from those of O_{2k}. Thus special attention is only required for the multiplicities of weights having k parts. These cases are most readily treated using difference characters.

If (λ) is a k-part partition then the difference character $[\lambda]''$ for SO_{2k} is defined as

$$[\lambda]''=[\lambda]_+-[\lambda]_- \qquad p_k=k. \tag{30}$$

It is convenient to write (King *et al* 1981)

$$[\lambda]''=[\square;\lambda/1^k]''=[\square;\mu]'' \tag{31}$$

where

$$[\mu]=[\lambda/1^k].$$

Noting that (El-Samra and King 1979)

$$[\square;\mu]''=\square''\langle\mu\rangle \tag{32}$$

where $\langle\mu\rangle$ is a character of Sp_{2k}, leads to

$$[\square;\mu]''=\mathcal{M}^{(\square;\boldsymbol{\sigma})}_{\square''\cdot(\boldsymbol{\tau})}M^{(\boldsymbol{\tau})}_{\langle\mu\rangle}(\square;\boldsymbol{\sigma})'' \tag{33}$$

and hence to

$$M^{(\square;\boldsymbol{\sigma})}_{[\square;\mu]''}=\mathcal{M}^{(\square;\boldsymbol{\sigma})''}_{\square''\cdot(\boldsymbol{\tau})}M^{(\boldsymbol{\tau})}_{\langle\mu\rangle} \tag{34}$$

with $\omega_\mu-\omega_\sigma$ being an *even* positive integer.

Making use of (28) and (30) yields

$$[\square;\mu]_\pm=\tfrac{1}{2}[M^{(\square;\boldsymbol{\sigma})}_{[\square;\mu]}(\square;\boldsymbol{\sigma})+M^{(\square;\boldsymbol{\sigma})''}_{[\square;\mu]''}(\square;\boldsymbol{\sigma})''] \tag{35}$$

and thence to

$$M^{(\square;\boldsymbol{\sigma})_{(-)^x}}_{[\square;\mu]_\pm}=\tfrac{1}{2}[M^{(\square;\boldsymbol{\sigma})}_{[\square;\mu]}\pm(-1)^x M^{(\square;\boldsymbol{\sigma})}_{[\square;\mu]''}]. \tag{36}$$

The multiplicities $M^{(\square;\boldsymbol{\sigma})}_{[\square;\mu]}$ are as for O_{2k} discussed in § 6. Thus we need only consider the evaluation of $M^{(\square;\boldsymbol{\sigma})}_{[\square;\mu]''}$ defined in (34).

The arrays $\mathcal{M}^{(\square;\boldsymbol{\sigma})}_{\square''\cdot(\boldsymbol{\tau})}$ are sparse. Consideration of the weights in $\square''\cdot(\boldsymbol{\tau})$ shows that the weight $(\square;\boldsymbol{\sigma})''$ arises if and only if

$$\{\boldsymbol{\tau}/2^s\}\supset\{\boldsymbol{\sigma}\} \tag{37}$$

and hence

$$\mathcal{M}^{(\square;\boldsymbol{\sigma})''}_{\square''\cdot(\boldsymbol{\tau})}=\delta^{(\boldsymbol{\sigma})}_{(\boldsymbol{\tau})} \qquad \text{if} \quad \omega_\tau=\omega_\sigma. \tag{38}$$

Furthermore, the non-null terms in $\mathcal{M}^{(\square;\sigma)}_{\square''\cdot(\tau)}$ are of sign

$$(-1)^{\frac{1}{2}(\omega_\tau-\omega_\sigma)\mathrm{mod}2} \tag{39}$$

In determining the terms that arise in $\square''\cdot(\tau)$ it is only necessary to consider the weights of $\square''$ of the form

$$[1^{k-2s}\bar{1}^{2s}]$$

where $(\bar{1})=(-1)$. We are thus led, simply and rapidly, to the evaluation of the few non-null entries for the array $\mathcal{M}^{(\square;\sigma)''}_{\square''\cdot(\tau)}$.

10. Explicitly k-dependent weight multiplicities

With the preceding results established it is now possible to determine the explicit k dependence of the weight multiplicities for any of the classical groups. By way of examples we give in table 1 the multiplicities $\mathcal{M}^{(\sigma)}_{(\omega)}$, for $\omega_\omega \leq 4$, as defined in equation (8). Where two entries occur, as in $\mathcal{M}^{(0)}_{(4)}$, the uppermost entry pertains to $N=2k$ and the lower to $N=2k+1$. These multiplicities may then be used in equation (12) to compute the k-dependent multiplicity numbers $M^{(\sigma)}_{\{\lambda\}}$ as given in table 2 for $\omega_\lambda \leq 4$. With these results established we may use (13*a*) and (13*b*) to compute the k dependence of the multiplicities $M^{(\sigma)}_{[\lambda]}$ and $M^{(\sigma)}_{\langle\lambda\rangle}$ for O_{2k} and Sp_{2k} respectively. Tables of

Table 1. Multiplicities $\mathcal{M}^{(\sigma)}_{(\omega)}$ for $\omega_\omega \leq 4$.

(ω) \ (σ)	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
(4)	1											0 1
(31)		1				0 1			1		0 1	
(2^2)			1						0 1			k
(21^2)				1			0 1		$k-1$	2 3	0 1	0 k
(1^4)					1			0 1		$k-2$	0 $k-1$	$\frac{1}{2}k(k-1)$
(3)						1						0 1
(21)							1		0 1		1 2	
(1^3)								1		0 1	$k-1$	0 k
(2)									1			0 1
(1^2)										1	0 1	k
(1)											1	0 1
(0)												1

Table 2. Multiplicities $M^{(\sigma)}_{\{\lambda\}}$ for $\omega_\lambda \leqslant 4$.

(σ)												
$\{\lambda\}$	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
$\{4\}$						0	0	0	k	k	0	$\frac{1}{2}k(k+1)$
	1	1	1	1	1	1	1	1	$(k+1)$	$(k+1)$	$(k+1)$	$\frac{1}{2}(k+1)(k+2)$
$\{31\}$						0	0	0	$2k-1$	$(3k-2)$	$3k$	$\frac{1}{2}k(3k-1)$
		1	1	2	3	1	2	3	$2k$	$3k$	$\frac{3}{2}k(k+1)$	$\frac{1}{2}(3k^2+3k+2)$
$\{2^2\}$							0	0	$(k-1)$	$2(k-1)$	0	k^2
			1	1	2	0	1	2	k	$(2k-1)$	$(2k-1)$	$k(k+1)$
$\{21^2\}$							0	0		$(3k-4)$	0	$\frac{3}{2}k(k-1)$
				1	3	0	1	3	$(k-1)$	$3(k-1)$	$3k-2$	$\frac{1}{2}k(3k-1)$
$\{1^4\}$								0			0	
					1	0	0	1	0	$(k-2)$	$(k-1)$	$\frac{1}{2}k(k-1)$
$\{3\}$									0	0	k	0
						1	1	1	1	1	$k+1$	$k+1$
$\{21\}$									0	0	0	0
$\{21\}$							1	2	1	2	1	$2k$
$\{1^3\}$										0		0
								1		1	$k-1$	k
$\{2\}$												k
									1	1		$k+1$
$\{1^2\}$											0	
										1	1	k
$\{1\}$												0
											1	1
$\{0\}$												1

these multiplicities have been given by King and Plunkett (1976) and will not be repeated here.

The spin multiplicities $\mathscr{M}^{(\Delta;\sigma)}_{\Delta\cdot(\tau)}$ for O_N follow from § 7 and are given in table 3 for $\omega_\tau \leqslant 4$. These multiplicities may be used in equation (19) to compute the multiplicities $M^{(\Delta;\sigma)}_{[\Delta;\lambda]}$. Again such tables have been given by King and Plunkett (1976). The multiplicities $M^{(\Delta;\sigma)_{(-)^x}}_{[\Delta;\lambda]_\pm}$ for SO_{2k} follow directly from equation (27) giving for example

$$M^{(\Delta;1)_-}_{[\Delta;31]_+} = \tfrac{1}{2}(k^3 - k^2 - 2k + 2).$$

The difference character multiplicities $\mathscr{M}^{(\square;\sigma)}_{\square''\cdot(\tau)}$ for $\omega_\tau \leqslant 4$ are given in table 4 and then used to derive the multiplicities $M^{(\square;\sigma)}_{[\square;\mu]''}$ shown in table 5. These latter multiplicities may be used in equation (36) to compute the weight multiplicities for the k-part tensor irreps of SO_{2k}.

It should be noted that for $\omega_\mu = \omega_\sigma$

$$M^{(\square;\sigma)}_{[\square;\mu]} = M^{(\square;\sigma)}_{[\square;\mu]''} = K^{(\sigma)}_{\{\mu\}} \tag{40}$$

and hence

$$M^{(\square;\sigma)_\pm}_{[\square;\mu]_\pm} = K^{(\sigma)}_{\{\mu\}} \qquad \text{and} \qquad M^{(\square;\sigma)_\mp}_{[\square;\mu]_\pm} = 0 \tag{41}$$

where $K^{(\sigma)}_{\{\mu\}}$ is an element of the Kostka matrix.

While it is a comparatively simple task to determine the k dependence of the difference weight multiplicities $M^{(\square;\sigma)}_{[\square;\mu]''}$ for SO_{2k} the k dependence for the weight

Table 3. Spin multiplicities $\mathcal{M}^{(\Delta;\sigma)}_{\Delta\cdot(\tau)}$ ($N = k$ or $2k+1$).

(τ) \ (σ)	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
(4)	1	—	—	—	—	1	—	—	—	—	—	—
(31)		1	—	—	—	$k-1$	1	—	$k-1$	—	—	—
(2^2)			1	—	—	—	1	—	—	1	—	—
(21^2)				1	—	—	$k-2$	3	$\frac{1}{2}(k-1)(k-2)$	$2k-4$	$\frac{1}{2}(k-1)(k-2)$	—
(1^4)					1	—	—	$k-3$	–	$\frac{1}{2}(k-2)(k-3)$	$\frac{1}{6}(k-1)(k-2)(k-3)$	$\frac{1}{24}k(k-1)(k-2)(k-3)$
(3)						1	—	—	1	—	—	—
(21)							1	—	$k-1$	2	$k-1$	—
(1^3)								1	—	$k-2$	$\frac{1}{2}(k-1)(k-2)$	$\frac{1}{6}k(k-1)(k-2)$
(2)									1	—	1	—
(1^2)										1	$k-1$	$\frac{1}{2}k(k-1)$
(1)											1	k
(0)												1

Table 4. Multiplicities $\mathcal{M}^{(\square;\sigma)}_{\square'\cdot(\tau)}$ for $\omega_\tau \leqslant 4$.

(τ) \ (σ)	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
(4)	1								-1			
(31)		1								-2		
(2^2)			1						$-k+1$			$\frac{1}{2}k(k-1)$
(21^2)				1						$-k+2$		
(1^4)					1							
(3)						1					$-k$	
(21)							1				$-k+1$	
(1^3)								1				
(2)									1			$-k$
(1^2)										1		
(1)											1	
(0)												1

Table 5. Multiplicities $M^{(\square;\sigma)}_{[\square;\mu]'}$ for $\omega_\mu \leqslant 4$.

(μ) \ (σ)	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
(4)	1	1	1	1	1				0	0		0
(31)		1	1	2	3				$k-1$	$k-1$		0
(2^2)			1	1	2				0	$k-1$		$\frac{1}{2}(k-1)$
(21^2)				1	3				$k-2$	$2k-4$		$\frac{1}{2}(k^2-3k+2)$
(1^4)					1				0	$k-3$		$\frac{1}{2}k(k-3)$
(3)						1	1	1				
(21)							1	2			$k-1$	
(1^3)								1			$k-2$	
(2)									1	1		0
(1^2)										1		$k-1$
(1)											1	
(0)												1

multiplicities $M^{(\square;\sigma)}_{[\square;\mu]}$ of O_{2k} is complicated by the complexity of the expansion of $[\square;\mu]$ into terms of the type $\square\cdot[\lambda]$ (King *et al* 1981).

11. Conclusions

In making physical applications of the classical Lie groups it is desirable to be able to see how the various group properties depend on the rank k of the group (Wybourne 1982). This avoids the need to engage in endless repetition each time a group of a different rank is considered. A knowledge of the rank dependence of group properties allows us to investigate general properties of representations without regard to specific ranks. In this paper we have outlined methods for determining the rank dependence of weight multiplicities extending the earlier work by King and Plunkett (1976).

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References

El-Samra N and King R C 1979 *J. Phys. A: Math. Gen.* **12** 2305
Freudenthal H 1954 *Indag. Math.* **16** 369
King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
King R C and Al-Qubanchi A H A 1981 *J. Phys. A: Math. Gen.* **14** 51
King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2059
King R C and Plunkett S P O 1976 *J. Phys. A: Math. Gen.* **9** 863
Konstant B 1959 *Trans. Am. Math. Soc.* **93** 53
Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)
Racah G 1964 *Group Theoretical Concepts and Methods in Elementary Particle Physics* ed F Gürsey (New York: Gordon and Breach) pp 1–36
Speiser D R 1964 *Group Theoretical Concepts and Methods in Elementary Particle Physics* ed F Gürsey (New York: Gordon and Breach) pp 201–76
Weyl H 1926 *Math. Z.* **24** 3
Wybourne B G 1982 *Rank dependency of group properties* in *Proc. 10th Int. Col. on Group-Theoretical Methods in Physics* to appear in *Physica*